§1 Introduction

review

k : NF or MLF $\to \bar{k}$: alg closure

$([k:\mathbb{Q}] < \infty)$ $([k:\mathbb{Q}_p] < \infty)$

$G_k := \mathrm{Gal}(\bar{k}/k)$

X : hyperbolic curve $/k$

$\pi_1((-))$: étale fund gp of $(-)$

$\rightsquigarrow \; 1 \to \pi_1(X \times_k \bar{k}) \to \pi_1(X) \to G_k \to 1$ (exact)

$\rightsquigarrow \; G_k \to \mathrm{Out}(\underbrace{\pi_1(X \times_k \bar{k})}_{\parallel \; \Pi_1})$ outer rep

inj?

$$
\begin{tikzcd}
G_k \arrow[r] \arrow[dr] \arrow[ddr] \arrow[dddr] \arrow[dddrr, bend right, "\text{injective (by Belyi)}"'] & \mathrm{Out}(\Pi_1) & \rightsquigarrow \text{inj !} \\
& \mathrm{Out}^{FC}(\Pi_1) \arrow[u, phantom, "\cup\!|" description] & \\
& \mathrm{Out}^{FC}(\Pi_2) \arrow[u, "\wr"] & \\
& \mathrm{Out}^{FC}(\Pi_3) \arrow[u, "\wr"] \arrow[r] & \mathrm{Out}(\pi_1(\mathbb{P}^1_k \setminus \{0,1,\infty\}))
\end{tikzcd}
$$

losure

(exact)

inj ?

<u>Thm</u> (Hoshi - Mochizuki)

$\mathrm{Out}^{FC}(\Pi_{n+1}) \to \mathrm{Out}^{FC}(\Pi_n)$ is inj $(\forall n \geq 1)$

Today,

<u>Thm</u> (Mochizuki)

$\mathrm{Out}^{FC}(\Pi_{n+1}) \to \mathrm{Out}^{FC}(\Pi_n)$ is bij $(\forall n \geq 4)$

$(\pi_1(\mathbb{P}^1_k \setminus \{0, 1, \infty\}))$

inj $(\forall n \geq 1)$

bij $(\forall n \geq 4)$

Rmk

$n \geq 4$, $(g, r) \notin \{(0,3), (1,1)\}$ cf. ([Nakamura]) (cf. [CombTopIII])

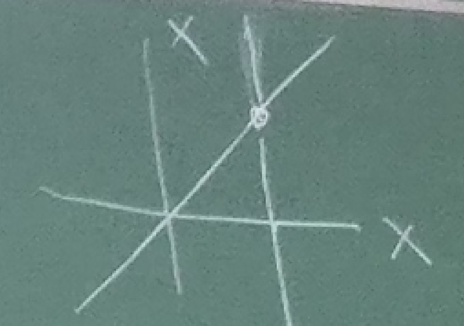
• affine case

$\mathrm{Out}^{Fc}(\Pi_n) \xrightarrow{\sim} \mathrm{Out}^{Fc}(\Pi_4) \overset{\text{Hoshi-Mochizuki}}{\overset{\sim}{\hookrightarrow}} \mathrm{Out}^{Fc}(\Pi_3) \overset{?}{\cong} \mathrm{Out}^{Fc}(\Pi_2) \overset{\text{Hoshi-Mochizuki}}{\cancel{\cong}} \hookrightarrow \mathrm{Out}^{Fc}(\Pi_1)$

• proper case

$\mathrm{Out}^{Fc}(\Pi_n) \xrightarrow{\sim} \mathrm{Out}^{Fc}(\Pi_4) \overset{?}{\cong} \hookrightarrow \mathrm{Out}^{Fc}(\Pi_3) \overset{?}{\cong} \hookrightarrow \mathrm{Out}^{Fc}(\Pi_2) \overset{\cancel{\cong}}{\hookrightarrow} \mathrm{Out}^{Fc}(\Pi_1)$

Notations



X : hyperbolic curve $/ \exists$ alg closed field $(ch = 0)$ of type (g, r)

g ← genus, r ← number of cusps

X_n : n-th configuration space

$:= \{(x_1, \dots, x_n) \in \underbrace{X \times_k \cdots \times_k X}_{n} \mid x_i \neq x_j \ (i \neq j)\}$

standard projections

$$X_n \to X_{n-1} \to \cdots \to X_2 \to X$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} \mapsto \cdots \mapsto \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto x_1$$

$$K_m := \mathrm{Ker}(\Pi_n \twoheadrightarrow \Pi_m)$$

$\Pi_n := \pi_1(X_n)$, $\Pi_0 = \{1\}$

$$\rightsquigarrow \Pi_n \twoheadrightarrow \Pi_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow \Pi_2 \twoheadrightarrow \Pi_1$$

$\alpha \in \mathrm{Aut}(\Pi_n)$

$\mathrm{Ker}(\Pi_n \twoheadrightarrow \Pi_{n'})$

α : F-admissible $\overset{def}{\iff}$ for $^\forall$ fiber subgroup $F \subseteq \Pi_n$, $\alpha(F) = F$

$\mathrm{Aut}^F(\Pi_n)$ $\cap$

$X_2 \to X$, $X_2 \to X$ $\rightsquigarrow$ $\Pi_2 \twoheadrightarrow \Pi_1$, $\Pi_2 \twoheadrightarrow \Pi_1$

α : C-admissible $\iff$ $\alpha(K_m) = K_m$ ($^\forall m$), and,

($\rightsquigarrow \alpha : K_m/K_{m+1} \xrightarrow{\sim} K_m/K_{m+1}$)

for $^\forall$cusp inertia subgp $I \subseteq K_m/K_{m+1}$

$\alpha(I)$, $\alpha^{-1}(I)$: cusp inertia subgp of K_m/K_{m+1}

$\mathrm{Aut}^C(\Pi_n)$ $\cap$

Today,

Thm (Mochi

Out

n) :=

standard projections

$(\Pi_n \twoheadrightarrow \Pi_m)$

$\twoheadrightarrow \Pi_{n'})$

$\subseteq \Pi_n$,

d,

)

K_{m+1}

K_m/K_{m+1}

$$Aut^{FC}(\Pi_n) := Aut^{F}(\Pi_n) \cap Aut^{C}(\Pi_n) \ni \text{FC-admissible}$$

$$Out^{FC}(\Pi_n) := Aut^{FC}(\Pi_n) \big/ Inn(\Pi_n)$$

Thus, the standard proj $X_{n+1} \xrightarrow{\Pi_{n+1} \twoheadrightarrow \Pi_n} X_n$ induces a hom $Out^{FC}(\Pi_{n+1}) \to Out^{FC}(\Pi_n)$

Rmk For any projection $X_{n+1} \to X_n$ of length 1

$Out^{FC}(\Pi_{n+1}) \to Out^{FC}(\Pi_n)$ coincide

Def

$$\begin{tikzcd}
Out^{FC}(\Pi_n) \arrow[r] & Out^{FC}(\Pi_1)
\end{tikzcd} \qquad \sigma \mapsto \sigma_1$$

$$\cup$$

$$Out^{FC}(\Pi_n)^{cusp} := \left\{ \sigma \in Out^{FC}(\Pi_n) \;\middle|\; \begin{array}{l} \sigma_1 \text{ induces the identity} \\ \text{permutation of the set of conj} \\ \text{classes of cusp inertia subgp} \subseteq \Pi_1 \end{array} \right\}$$

§2 proof

Prop 1

X

Im (

Prop 2

-admissible

$\twoheadrightarrow \Pi_1$

induces

length 1

cide

the identity

f the set of conj

inertia subgp $\subseteq \Pi_1$

$\boxed{\S 2}$ proof

Outline

Prop 1 X : affine ($\Leftrightarrow r \geq 1$)

$$\mathrm{Im}\left(\mathrm{Out}^{\mathrm{FC}}(\Pi_3) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_2)\right) \supseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_2)^{\Delta +}$$

Prop 2 X : arbitrary, $n \geq 4$

$$\mathrm{Im}\left(\mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_n)\right) \supseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\Delta +}$$

Prop 3 X : arbitrary $n \geq 4$

$$\mathrm{Im}\left(\mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_n)\right) \supseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}}$$

Thm X : arbitrary, $n \geq 4$

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}) \twoheadrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$$

Rmk X_n, Π_n

$n=4$

) "GT" arises ← 2nd configuration sp of tripod

Grothendieck - Teichmüller gp

3

) a tripod arises

2

) a cusp arises

1

prop3 ⇝ Thm

lem1 $n \geq 1$

The composite

$$Out^{FC}(\Pi_n) \to Out^{FC}(\Pi_1) \to \mathfrak{S}_r$$

is surjective

Considering the action on the set of conj classes of cusp inertia subgps $\subseteq \Pi_1$

symmetric gp on r letters

Def $Out^{FC}(\Pi$

$\cup$

$Out^{FC}(\Pi$

Pf of (